

State Space Control and Observation of DC Motor Dynamics and Vehicle Lane - Keeping Systems

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Abstract—This project presents the design and simulation of state - space control strategies for two distinct dynamic systems: a DC Motor and a lateral vehicle lane - keeping system. In the first part, a DC motor is modeled and analyzed for controllability and observability. A state-feedback controller is implemented, augmented with an integral term to ensure zero steady-state error and robust disturbance rejection. A Luenberger observer is further designed to estimate unmeasured states, facilitating output-feedback control. The second part addresses the non-linear kinematics of a vehicle for lane-keeping applications. Through linearization around the equilibrium point and pole-placement techniques, a robust linear controller is developed. Simulation results in MATLAB/Simulink environment demonstrate that both systems achieve asymptotic stability, with the DC motor providing precise speed tracking and the vehicle system maintaining lane centering across a range of initial conditions despite non-linear effects.

Index Terms—State Space Control, DC Motor, Lane - Keeping Systems, Luenberger Observer, Pole Placement, Integral Control, Robustness, MATLAB, Simulink

I. DC MOTOR CONTROL

A. Calculation Of Matrices A, B, C and D

To find matrices A, B, C , and D , we first transition to the state-space representation using the standard equations:

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (1)$$

$$y(t) = Cx(t) + Du(t) \quad (2)$$

From the differential equations describing the system, which are listed below, we observe that the variables being differentiated are the current i and the angular velocity ω :

$$\frac{di}{dt} = -\frac{R}{L}i - \frac{K_e}{L}\omega + \frac{1}{L}u \quad (3)$$

$$\frac{d\omega}{dt} = \frac{K_t}{J}i - \frac{b}{J}\omega + \frac{1}{J}d \quad (4)$$

Thus, the state vector is defined as:

$$x = \begin{bmatrix} i \\ \omega \end{bmatrix}$$

The system input is the voltage u . Since the disturbance d for question A is considered to be zero, the input is simply:

$$u = u$$

The system output is: $y = \omega$.

By substituting the above into the system equations, we obtain the following:

$$\frac{di}{dt} = -\frac{R}{L}i - \frac{K_e}{L}\omega + \frac{1}{L}u \quad (5)$$

$$\frac{d\omega}{dt} = \frac{K_t}{J}i - \frac{b}{J}\omega \quad (6)$$

These can be written in matrix form (state-space representation) as follows:

$$\begin{bmatrix} \frac{di}{dt} \\ \frac{d\omega}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{K_e}{L} \\ \frac{K_t}{J} & -\frac{b}{J} \end{bmatrix} \begin{bmatrix} i \\ \omega \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u \quad (7)$$

$$y = [0 \quad 1] \begin{bmatrix} i \\ \omega \end{bmatrix} + [0]u \quad (8)$$

Therefore, the matrices are:

$$A = \begin{bmatrix} -\frac{R}{L} & -\frac{K_e}{L} \\ \frac{K_t}{J} & -\frac{b}{J} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}, \quad C = [0 \quad 1] \quad \text{and} \quad D = 0 \quad (9)$$

Substituting the constants with the values $J = 0.01$, $b = 0.1$, $K_t = K_e = 0.01$, $R = 1$, $L = 0.5$ we obtain:

$$A = \begin{bmatrix} -2 & -0.02 \\ 1 & -10 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad (10)$$

and obviously $C = [0 \quad 1]$ and $D = 0$.

B. Controllability Check

To see whether the above Linear Time-Invariant (LTI) system is controllable, we must construct its controllability matrix and check if it is full rank.

Since our system has 2 state variables, it will be of the 2nd order. Thus, its controllability matrix will be:

$$C = [B \quad AB]$$

From the previous part, we calculated that $A = \begin{bmatrix} -2 & -0.02 \\ 1 & -10 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$$\Rightarrow AB = \begin{bmatrix} -2 & -0.02 \\ 1 & -10 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

Therefore, the controllability matrix will be:

$$C = \begin{bmatrix} 2 & -4 \\ 0 & 2 \end{bmatrix}$$

Finally, for the system to be controllable, the determinant of matrix C must be non-zero, so that C is full rank.

We calculate the determinant as follows:

$$\det(C) = \begin{vmatrix} 2 & -4 \\ 0 & 2 \end{vmatrix} = 2 \cdot 2 - (-4) \cdot 0 = 4$$

Since the determinant of the controllability matrix is non-zero, the matrix has full rank, and therefore the system is controllable.

C. Finding Equilibrium Points

To determine the equilibrium points of a system in the state space, we must locate the points where the state variables do not change with time. That is, the points where:

$$\dot{x}(t) = 0$$

Let $x_e = \begin{bmatrix} i_e \\ \omega_e \end{bmatrix}$ be an equilibrium point resulting from the application of a constant input u_e .

Setting $\dot{x} = 0 \Rightarrow Ax_e + Bu_e = 0$ and using the matrices calculated above, we find:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 & -0.02 \\ 1 & -10 \end{bmatrix} \begin{bmatrix} i_e \\ \omega_e \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u_e$$

$$\Rightarrow 0 = -2i_e - 0.02\omega_e + 2u_e \quad \text{and} \quad 0 = i_e - 10\omega_e$$

From the above, we conclude that the points which can potentially constitute equilibrium points are those that satisfy the relationship:

$$i_e = 10\omega_e$$

Geometrically, this means that it is not possible for any random current-speed pair to constitute an equilibrium point, unless this point belongs to the line $i = 10\omega$. Any such point can be achieved by applying an appropriate, constant excitation voltage to the system.

D. State-Space Controller Design

We want the controller to force the motor speed to follow a predetermined reference value r . To achieve this, we use a State Feedback Controller with Feedforward/Reference Gain.

Initially, the control law is as follows: $u = -Kx + k_r r$, where $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ is the feedback gain matrix, which we calculate by choosing where we want to place the poles of the closed-loop system so that it is stable and converges quickly. k_r is the reference gain (Feedforward Gain), which ensures that in the steady state, the error will be as close to zero as possible.

Substituting the control law $u = -Kx + k_r r$ into the system equation, we obtain:

$$\dot{x} = (A - BK)x + Bk_r r$$

and

$$y(t) = Cx$$

To have zero steady-state error, we calculate the reference gain as follows:

$$k_r = \frac{-1}{C(A - BK)^{-1}B} \quad (11)$$

Operation of State Feedback Controller with Feedforward/Reference Gain:

- **State Feedback:** In contrast to PID controllers that only measure the output (in this case, the speed error), the controller we use reads all the state variables of the system (in this case, the current i and the speed ω). Therefore, by multiplying the state vector x by the gain matrix K , we can move the poles of the closed-loop system exactly where we want, fully controlling its dynamics.
- **Feedforward/Reference Gain:** Simple state feedback on its own stabilizes the system but does not guarantee that the speed will reach exactly the reference value. The gain k_r is specifically calculated to scale the input so that in the steady state, the motor output exactly equals the desired speed.

E. Simulation of the Closed-Loop System Using MATLAB

We choose—arbitrarily—the poles to be located at -10 and -15 .

The step response of the motor is shown below, obtained by creating a closed loop using the state feedback controller, which forces the motor to follow the reference value r . By choosing a reference $r = 1$, the following plot was designed, from which we observe that indeed, due to the gain k_r , the motor output converges to the desired value (the reference value), while the system remains stable. The absence of overshoot is attributed to the fact that the poles were chosen to be real numbers ($\text{Im}\{\text{poles } 1 \text{ and } 2\} = 0$).

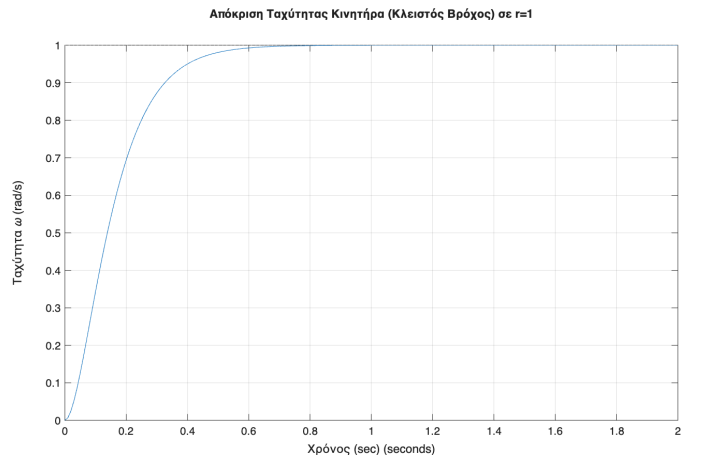


Fig. 1. Motor Step Response with State Feedback Controller

F. Finding the Transfer Function from Disturbance to Output - Bode Diagram Design

To find the transfer function from the disturbance d to the output, it is necessary to incorporate the disturbance into the state-space model while simultaneously setting the desired reference to zero, so that we may observe only the effect of the disturbance.

We use the system equations:

$$\begin{aligned}\frac{di}{dt} &= -\frac{R}{L}i - \frac{K_e}{L}\omega + \frac{1}{L}u \\ \frac{d\omega}{dt} &= \frac{K_t}{J}i - \frac{b}{J}\omega + \frac{1}{J}d\end{aligned}$$

Of these, the disturbance appears only in $\frac{d\omega}{dt} = \frac{K_t}{J}i - \frac{b}{J}\omega + \frac{1}{J}d$. Thus, taking the disturbance into account, the system in state-space is written as follows:

$$\dot{x} = Ax + Bu + B_d d \quad (12)$$

The matrix B_d is the matrix that introduces the disturbance into the state equation. Using the equation $\frac{1}{J} = \frac{1}{0.01} = 100$, we calculate it as: $B_d = \begin{bmatrix} 0 \\ 100 \end{bmatrix}$.

Since we want to find the transfer function from the disturbance to the output, by setting $r = 0$, we have: $u = -Kx$. Substituting u into the state equation, we get:

$$\dot{x} = Ax + B(-K)x + B_d d \Rightarrow \dot{x} = (A - BK)x + B_d d$$

For simplicity, we define the matrix $A_{cl} = A - BK$, and thus we have:

$$\begin{aligned}\dot{x} &= A_{cl}x + B_d d \\ y &= Cx\end{aligned}$$

Finally, we calculate the transfer function as follows:

$$G_{yd}(s) = \frac{Y(s)}{D(s)} = C(sI - A_{cl})^{-1}B_d \quad (13)$$

By using the poles at positions -15 and -10 , we find the matrix A_{cl} , which for brevity we provide below:

$$\begin{aligned}A_{cl} &= \begin{bmatrix} -15 & 0 \\ 1 & -10 \end{bmatrix} \\ \Rightarrow G_{yd}(s) &= [0 \quad 1] \begin{bmatrix} s+15 & 0 \\ -1 & s+10 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 100 \end{bmatrix} \\ \Rightarrow G_{yd}(s) &= \frac{100(s+15)}{(s+15)(s+10)} \\ \Rightarrow G_{yd}(s) &= \frac{100}{s+10} \quad (14)\end{aligned}$$

The Bode diagram for the closed-loop system is shown below, from which we conclude that the closed-loop system behaves like a low-pass filter against disturbances. It allows slow disturbances to pass through while rejecting fast ones. To correct this issue and ensure that the magnitude curve goes into negative dB even at low frequencies, an integral term (I) will be required.

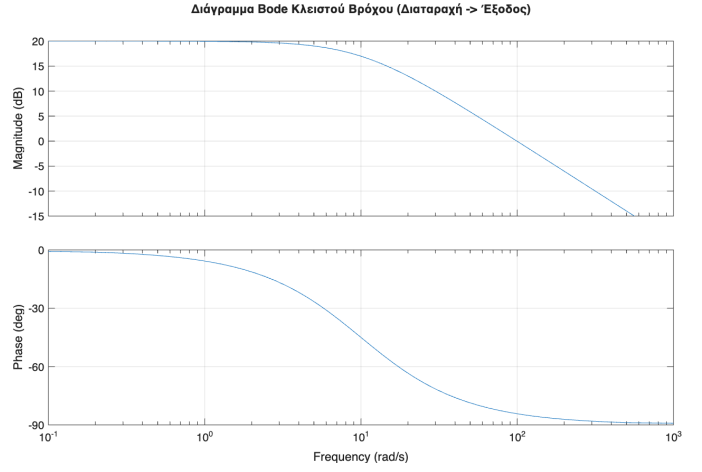


Fig. 2. Bode Diagram of the Disturbance-to-Output Transfer Function

G. Integral Controller Design - Bode Diagram Design

To eliminate steady-state error despite the presence of a non-zero constant disturbance, we add an integral term to the controller designed above. In state-space, this is achieved by incorporating the integral of the error into the state equation as a new state variable.

Initially, we define the error as: $e = r - y = r - Cx$. Since the new variable will be the integral of the error e , it follows that its derivative will be exactly equal to the error.

$$\Rightarrow \dot{x}_i = r - Cx \quad (15)$$

Using this new variable, we create the augmented state vector:

$$x_{aug} = \begin{bmatrix} x \\ x_i \end{bmatrix}$$

Thus, the new system is written as:

$$\begin{bmatrix} \dot{x} \\ \dot{x}_i \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_i \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r \equiv \dot{x}_{aug} = A_{aug}x_{aug} + B_{aug}u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r \quad (16)$$

Using the matrices $A = \begin{bmatrix} -2 & -0.02 \\ 1 & -10 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $C = [0 \quad 1]$, we calculate:

$$A_{aug} = \begin{bmatrix} -2 & -0.02 & 0 \\ 1 & -10 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad \text{and} \quad B_{aug} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

The new control law will be: $u_{aug} = -Kx_{aug} = -Kx - K_i x_i$, where $K = [K \quad K_i]$.

Finally, the transfer function of the augmented system, which now includes the integral of the error, is given by the following formula:

$$G_{yd,aug}(s) = C_{aug}(sI - A_{cl})^{-1}B_{d,aug} \quad (17)$$

For brevity, we provide the transfer function of the augmented system, having chosen the poles to be located at $p_1 = -10$, $p_2 = -15$ and $p_3 = -20$:

$$G_{yd,aug}(s) = \frac{100s(s+35)}{s^3 + 45s^2 + 650s + 3000} \quad (18)$$

Below is the Bode plot of the closed-loop transfer function from disturbance to output. From the diagram, we conclude that at very low frequencies (near 0), the curve drops to minus infinity (practically a very large negative value in absolute terms). This means that a constant disturbance is completely attenuated. The gain is zero, so the steady-state error is also zero. At intermediate frequencies, the curve rises, reaches a peak, and then falls again. The system now behaves like a band-pass filter against disturbances.

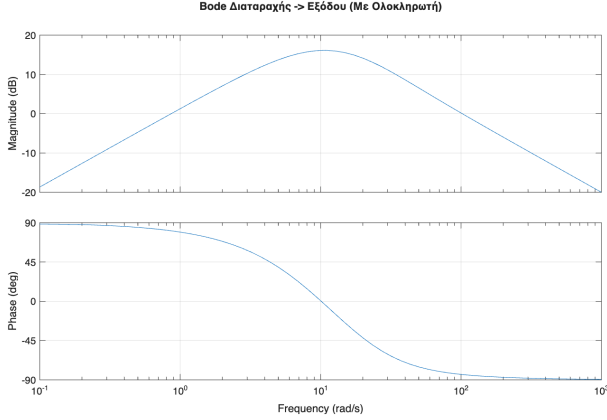


Fig. 3. Bode Diagram (With Integrator) of the Disturbance-to-Output Transfer Function

H. Finding the "Worst-Case" Frequency

For a sinusoidal disturbance of unit amplitude: $d(t) = 1 \sin(\omega t)$, the "worst-case" frequency is the one that causes the largest oscillation at the motor output. That is, the largest error. This frequency is found at the peak of the magnitude curve in the Bode diagram. Using MATLAB and the commands characteristics and peak response, we found the worst-case frequency to be equal to 10.9.

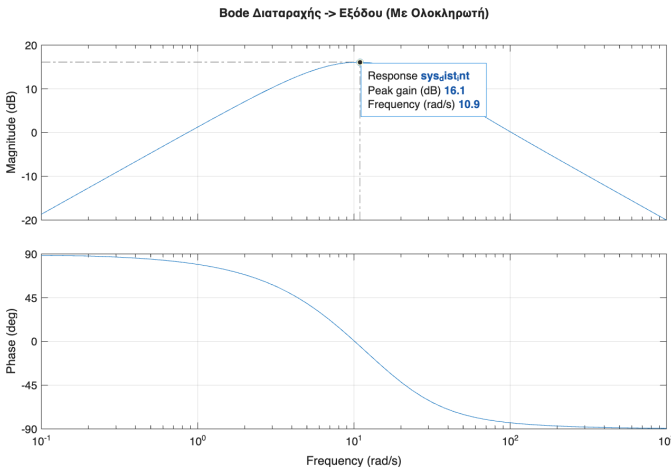


Fig. 4. Bode Magnitude Plot highlighting the resonance peak at 10.9 rad/s

I. State Observer Design

In reality, we often do not have sensors available to monitor the values of all variables. Since we only measure the speed ω , there is a need for a mathematical estimation of the current i . This is precisely the role of the State Observer.

The observer is essentially a mathematical simulation of the motor running in parallel with the physical one, and it is described by the equation:

$$\dot{\hat{x}}(t) = A\hat{x} + Bu + L(y - \hat{y}) \quad (19)$$

where $\hat{x} = \begin{bmatrix} \hat{i} \\ \hat{\omega} \end{bmatrix}$ are the estimated states and $L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$ is the observer gain matrix.

For the observer to function correctly, it must be faster than the controller so that it provides information before the controller has time to react. The estimation error is defined as: $e = x - \hat{x}$. To ensure it reaches zero quickly, the dynamics of which are defined by the term $(A - LC)$, an appropriate matrix L must be chosen such that its eigenvalues (the observer poles) are negative.

By arbitrarily choosing the poles at $p_{obs} = [-30, -40]$, we calculate matrix L as follows:

$$A - LC = \begin{bmatrix} -2 & -0.02 \\ 1 & -10 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$\Rightarrow A - LC = \begin{bmatrix} -2 & -0.02 - l_1 \\ 1 & -10 - l_2 \end{bmatrix}$$

$$\Rightarrow \det(sI - (A - LC)) = s^2 + (12 + l_2)s + (20.02 + 2l_2 + l_1)$$

We equate the polynomial resulting from the above determinant with the desired one obtained from our chosen poles:

$$\Rightarrow s^2 + (12 + l_2)s + (20.02 + 2l_2 + l_1) = (s - (-30))(s - (-40))$$

$$\Rightarrow s^2 + (12 + l_2)s + (20.02 + 2l_2 + l_1) = s^2 + 70s + 1200$$

Equating the coefficients, we obtain:

$$\Rightarrow L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} 1064 \\ 58 \end{bmatrix} \quad (20)$$

J. Controller Design with Output Feedback Using an Observer

For the design of the controller with output feedback, we use the control law as follows:

$$u = -K\hat{x} + k_r r \quad (21)$$

In the state-space with an observer, the controller itself becomes a complete dynamic system. By substituting u from the control law into the observer equation, we have:

$$\dot{\hat{x}} = A\hat{x} + B(-K\hat{x} + k_r r) + L(y - C\hat{x})$$

$$\Rightarrow \dot{\hat{x}} = (A - BK - LC)\hat{x} + Ly + Bk_r r$$

The system of the two equations below constitutes the controller:

$$\dot{\hat{x}} = (A - BK - LC)\hat{x} + Ly + Bk_r r \quad (22)$$

$$u = -K\hat{x} + k_r r \quad (23)$$

It receives the actual speed y and the reference r as inputs, calculates through the variable \hat{x} , and provides the appropriate command u to the motor.

K. Controller Simulation in Simulink

Using Simulink, the following layout was designed, which simulates the operation of the controller with output feedback using an observer.

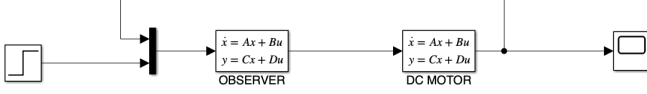


Fig. 5. Simulink block diagram representing the output feedback control system with state observer

After running the simulation, we obtained the following graphical representation:

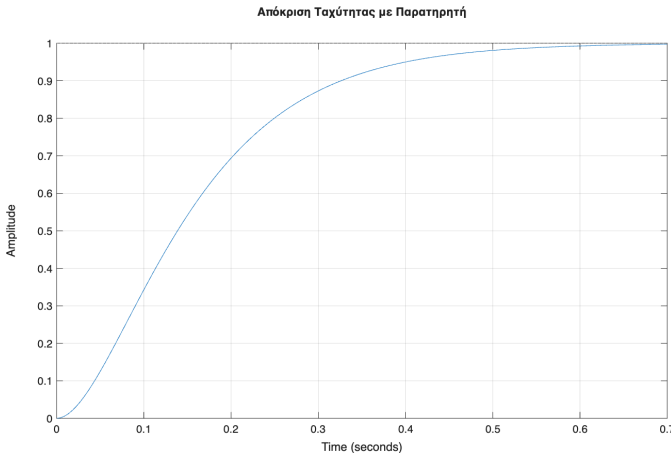


Fig. 6. System response under output feedback control with state estimation

Based on the graphical representation of the speed response, the following conclusions can be drawn. Initially, we observe that the system tracks the unit step input ($r = 1$) perfectly with zero steady-state error, confirming the correct calculation of the feedforward gain ($k_r = 75$). Regarding the dynamic characteristics, the transient response is exceptionally smooth, with absolutely no overshoot, while the settling time ranges within satisfactory levels (approximately 0.6 to 0.7 seconds).

II. TRAFFIC LANE KEEPING CONTROL

A. State Feedback Controller Design

To design the control system that will keep the following vehicle at the center of the traffic lane, we consider the following equations:

$$\dot{x} = V \cos \theta \quad (24)$$

$$\dot{y} = V \sin \theta \quad (25)$$

$$\dot{\theta} = Vu \quad (26)$$

Where $V = 1$ is the vehicle speed, which we assume remains constant, x and y are the vehicle coordinates, and θ is the angle formed with the horizontal axis (x). The controlled input u represents the steering wheel angle.

Since the goal is to keep the vehicle at the center of the lane, the longitudinal displacement x does not affect the lateral position control within the lane. What interests us is the lateral displacement y and the vehicle's heading angle θ . Therefore, the state variable vector will be:

$$X = \begin{bmatrix} y \\ \theta \end{bmatrix}$$

To maintain the center of the lane, the desired equilibrium conditions are:

$$y_e = 0, \quad \theta_e = 0, \quad u_e = 0$$

Thus, the system equations are:

$$\begin{aligned} \dot{y} &= \sin \theta \\ \dot{\theta} &= u \end{aligned}$$

Linearizing the equations around the equilibrium point ($y = \theta = 0$), we obtain:

$$\begin{aligned} \dot{y} &= \theta \\ \dot{\theta} &= u \end{aligned}$$

For very small angles, we know from the Taylor series expansion that: $\sin \theta \approx \theta$.

$$\Rightarrow \begin{bmatrix} \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ \theta \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad (27)$$

where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

From the control law:

$$u = -KX = - \begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} y \\ \theta \end{bmatrix} = -k_1 y - k_2 \theta$$

The closed-loop system matrix is: $A_{cl} = A - BK$

$$\Rightarrow A_{cl} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}$$

$$\Rightarrow \det(sI - A_{cl}) = \begin{vmatrix} s & -1 \\ k_1 & s + k_2 \end{vmatrix} = s(s + k_2) - (-1)k_1$$

$$\Rightarrow \det(sI - A_{cl}) = s^2 + k_2 s + k_1 = 0 \quad (28)$$

In order for the vehicle to remain in the lane, the system must be asymptotically stable. This means that the roots of the polynomial $s^2 + k_2 s + k_1$ must have a negative real part. Initially, we choose the poles: $p_1 = -1$ and $p_2 = -2$.

$$\Rightarrow (s - p_1)(s - p_2) = (s + 1)(s + 2) = s^2 + 3s + 2 = 0$$

$$\Rightarrow s^2 + k_2 s + k_1 = s^2 + 3s + 2 = 0$$

Equating the coefficients, we obtain: $k_1 = 2$ and $k_2 = 3$.

Therefore, a suitable controller to maintain the vehicle within the lane is:

$$u = -KX = - \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} y \\ \theta \end{bmatrix} = -2y - 3\theta \quad (29)$$

B. Simulink Simulation of the Closed-Loop System

The simulation equations are:

$$\dot{y} = \sin \theta \quad (30)$$

$$\dot{\theta} = -k_1 y - k_2 \theta \quad (31)$$

Using Simulink, the following configuration was designed to simulate the operation of the closed-loop system:

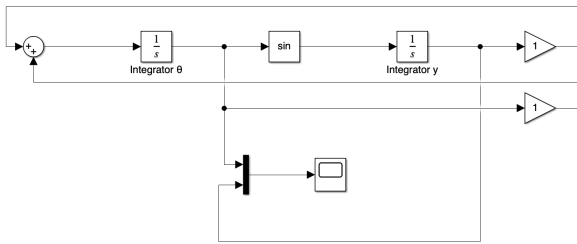


Fig. 7. Simulink block diagram for the lane-keeping control system.

Note: Due to a typographical error in the initial diagram, the summer block has incorrect signs (it should have been - and -), and the gains currently have unit values instead of the calculated values of 2 and 3.

After running the simulation (with the correct summer signs and gain values of 2 and 3), we obtained the following graphical representation:

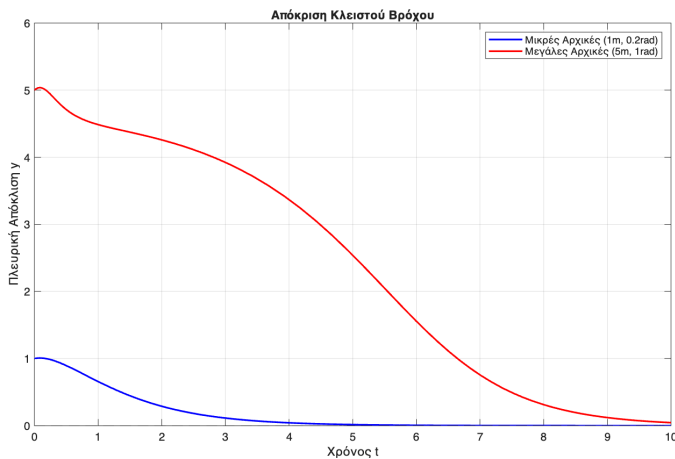


Fig. 8. Closed-loop response for different initial conditions (small vs. large deviations)

The graphical representation above confirms the asymptotic stability of the closed-loop system, as both trajectories eventually converge to the center of the lane ($y = 0$). We observe that

for small initial deviations (blue line), the response is smooth and direct, whereas for larger initial values (red line), there is a brief initial increase in the deviation before the final recovery. This highlights the influence of non-linear terms and the ability of the linear controller to remain robust even in conditions far from the linearization point.

In conclusion, the controller achieves its goal for a wide range of initial values, eliminating the position error without exhibiting permanent oscillations or instability.

For which initial values does the vehicle return to the center of the lane?: For the linearized system (based on the assumption $\sin \theta \approx \theta$), which is asymptotically stable, the system tends toward the point $(0, 0)$ for any initial conditions. Therefore, it returns to the center of the lane.

For the actual, non-linear system, the controller is guaranteed to work correctly only near the operating point—that is, for small initial angles.

III. CONCLUSION

This project successfully demonstrated the design, application and simulation of state - space control strategies for two distinct dynamic systems. A DC Motor and a lateral vehicle lane - keeping system. Through the use of the MATLAB/Simulink environment, both systems were shown to achieve asymptotic stability while fulfilling their specific operational objectives

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