Electrostatic Grating of Nanorods

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***Abstract*—This paper investigates the electrostatic field and potential distribution within an infinite, one-dimensional periodic array of dielectric nanorods subjected to a uniform transverse electric field. By applying rigorous boundary conditions and utilizing complex Laurent series expansions, closed-form analytical solutions for the scattered and internal potentials are derived using a dipole approximation. The model accounts for mutual coupling between adjacent cylinders through a geometric lattice sum factor that dictates the non-locality of the array. The analytical solutions are supported by graphical representations of the electric and potential fields across the xy-plane, evaluating the effects of varying dielectric permittivities, angles of field incidence, and inter-cylinder periodicities. Results indicate that the geometric packing density strictly controls inter-rod interactions; dense arrays exhibit significant field distortion and strong coupling, whereas sparse configurations smoothly revert to isolated cylinder behavior.**

***Index Terms*—Boundary Value Problems, Dielectric Cylinders, Electrostatic Scattering, Nanorods.**

I. Problem Formulation

Consider a cylinder of radius a with a dielectric constant $\varepsilon > 1$, with its axis parallel to the $\hat{z}$-axis. The applied field is given by:

$$E_{\text{back}} = E_0(\hat{x}\cos\theta + \hat{y}\sin\theta).$$

In the xy-plane, the potential outside and inside the cylinder is written in polar coordinates (r, ϕ) around its center:

- Outside (vacuum): keeping only the $m = 1$ harmonic (the only one excited by a homogeneous field):

$$V_{\text{out}}(r,\phi) = -E_0 r\cos(\phi-\theta) + A\frac{a^2}{r}\cos(\phi-\theta).$$

- Inside (material with permittivity ε):

$$V_{\text{in}}(r,\phi) = Br\cos(\phi-\theta).$$

II. Boundary Conditions and Solution

We impose boundary conditions at $r = a$:

- Continuity of potential: $V_{\text{out}} = V_{\text{in}}$,
- Continuity of normal displacement field ($\hat{n}\cdot D$): $\varepsilon_0\partial_r V_{\text{out}} = \varepsilon\varepsilon_0\partial_r V_{\text{in}}$.

Solving yields:

$$A = E_0\frac{\varepsilon-1}{\varepsilon+1} \equiv \xi E_0 \quad \text{and} \quad B = -\frac{2}{\varepsilon+1}E_0$$

Therefore, we have:

$$V_{\text{out}} = -E_0 r\cos(\phi-\theta) + \xi E_0\frac{a^2}{r}\cos(\phi-\theta)$$
$$V_{\text{in}} = -\frac{2}{\varepsilon+1}E_0 r\cos(\phi-\theta)$$
$$(\text{using } E = -\nabla V)$$

III. Form of Total Scattered Electrostatic Potential and Potential Inside the Cylinder

The centers of the cylinders are located at $(0, 2nb)$ with $n \in \mathbb{Z}$. For each cylinder, we define local polar coordinates (r_n, ϕ_n).

Assuming, based on the problem statement, that the presence of the other cylinders does not introduce higher azimuthal harmonics (we restrict our analysis to $m = 1$ for each cylinder), the total scattered potential is written as a superposition of dipoles for each cylinder. This uses the same coefficients across the entire array, but with local coordinate angles:

$$V_{\text{scat}}(x,y) = \sum_{n\in\mathbb{Z}}\frac{a^2}{r_n}[A_x\cos\phi_n + A_y\sin\phi_n] \tag{1}$$

where $\cos\phi_n = \frac{x}{r_n}$, $\sin\phi_n = \frac{y-2nb}{r_n}$, and $r_n = \sqrt{x^2+(y-2nb)^2}$.

The internal potential within each cylinder remains linear. Therefore, for the reference cylinder (or any cylinder n):

$$V_{\text{in}}^{(0)}(r,\phi) = B_x r\cos\phi + B_y r\sin\phi$$
$$\Downarrow$$
$$V_{\text{in}}^{(0)}(x,y) = B_x x + B_y y$$

IV. Laurent Series

We use the complex representation $z = re^{i\phi} = x + iy$ and centers at $z_n = i2nb$.

For $|z| < |z_n|$ the following holds:

$$\frac{1}{z-z_n} = -\frac{1}{z_n}\sum_{m=0}^{\infty}\left(\frac{z}{z_n}\right)^m$$

Taking the real and imaginary parts and noting that:

$$\frac{\cos\phi_n}{r_n} = \Re\left(\frac{1}{z-z_n}\right) \quad \text{and} \quad \frac{\sin\phi_n}{r_n} = \Im\left(\frac{1}{z-z_n}\right)$$

we arrive, for $r < 2|n|b$, at series of the form:

$$\frac{\cos\phi_n}{r_n} = \sum_{m=1}^{\infty} C_{m,n} r^{m-1}\cos(m\phi) + D_{m,n} r^{m-1}\sin(m\phi)$$
$$\frac{\sin\phi_n}{r_n} = \sum_{m=1}^{\infty} \tilde{C}_{m,n} r^{m-1}\cos(m\phi) + \tilde{D}_{m,n} r^{m-1}\sin(m\phi)$$

with coefficients $C_{m,n}, D_{m,n} \propto (2|n|b)^{-(m+1)}$ and alternating signs due to $\arg z_n = \pm\pi/2$.

The Laurent representation $\sim r^{-m}$ arises if we expand for $r > 2|n|b$.

Both forms are consistent and useful depending on the region of convergence; here we need the expansion for $r < a < b$.

V. Application of Boundary Conditions to Find the Unique Solution

Due to periodicity, all cylinders share the same coefficients (A_x, A_y) outside and (B_x, B_y) inside. The presence of the rest changes the local field at the center of each cylinder:

- The x-component is reduced
- The y-component is enhanced

After summing the fields at the center with two symmetric neighbors per n, a closed form emerges with a known lattice sum:

$$S = 2a^2 \sum_{n=1}^{\infty} \frac{1}{(2nb)^2} = \frac{a^2}{b^2}\frac{\pi^2}{12} \tag{2}$$

Setting $\xi = \frac{\varepsilon-1}{\varepsilon+1}$ (and $\xi \to 1$ for PEC) and writing $E_{0x} = E_0\cos\theta$ and $E_{0y} = E_0\sin\theta$, the self-consistent system yields:

$$A_x = \frac{\xi}{1+\xi S}E_{0x} \quad \text{and} \quad A_y = \frac{\xi}{1-\xi S}E_{0y} \tag{3}$$

The internal potential in each cylinder is:

$$V_{\text{in}}^{(n)}(x,y) = B_x x + B_y(y-2nb),$$
$$B_x = -\frac{2}{\varepsilon+1}(E_{0x} - SA_x),$$
$$B_y = -\frac{2}{\varepsilon+1}(E_{0y} + SA_y)$$

The total potential at any point (x, y) (excluding the interior of a specific cylinder) is:

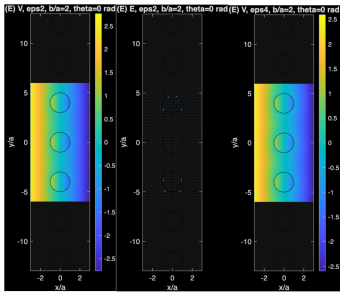
$$V(x,y) = -E_{0x}x - E_{0y}y + \sum_{n\in\mathbb{Z}} a^2\left[A_x\frac{x}{r_n^2} + A_y\frac{y-2nb}{r_n^2}\right] \tag{4}$$

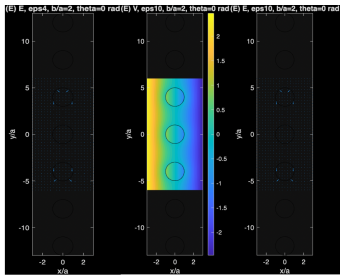
$$r_n^2 = x^2 + (y-2nb)^2$$

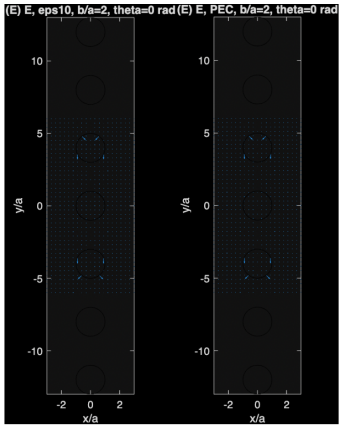
while inside a cylinder, we replace the n-th term with the linear $V_{\text{in}}^{(n)}$ for smoothness and application of the interface conditions. The field is $E = -\nabla V$. The above solution satisfies the boundary conditions (continuity of V and $\hat{n}\cdot D$) under the assumption $m = 1$.

VI. Plotting of the Electric and Potential Fields in the X-Y Plane (1)

In the following plots, the electric and potential fields in the X-Y plane are depicted for 3 materials with permittivities $\varepsilon = 2, 4, 10$, as well as for perfectly conducting cylinders. The assumptions are: $b = 2a$ and $\theta = 0$.



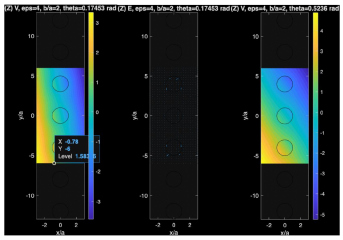


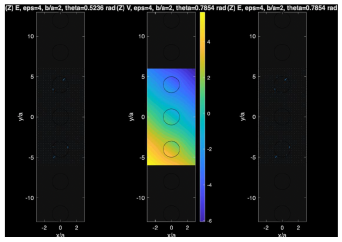


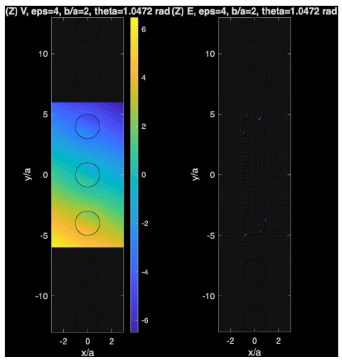
As we observe from the graphs above, for $\theta = 0$, the field is horizontal. By increasing the permittivity ε, i.e., for a non-PEC (Perfect Electric Conductor) material, the external term $A_x = \frac{\xi}{1+\xi S}E_0$ approaches the PEC limit ($\xi \to 1$), and the field slips around the cylinders with increasingly stronger distortion.

VII. Plotting of the Electric and Potential Fields in the X-Y Plane (2)

In the following plots, the electric and potential fields in the X-Y plane are depicted for angles $\theta = 10°, 30°, 45°, 60°$. The assumptions are: $b = 2a$ and $\varepsilon = 4$.



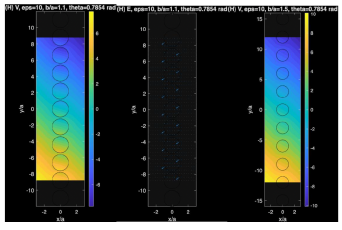


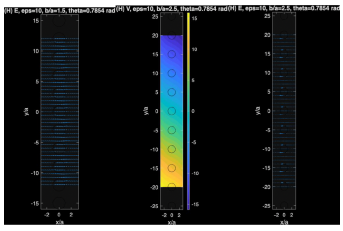


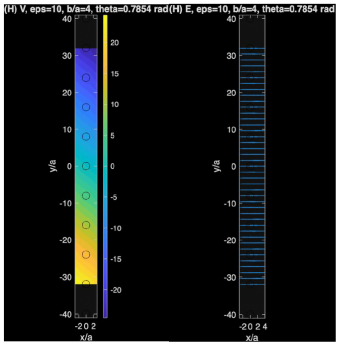
As we observe from the plots above, the projection along the y-axis excites the enhanced branch $A_y = \frac{\xi}{1-\xi S} E_{0y}$. As the angle θ increases, the equipotential curves tilt and a stronger deflection appears along the direction of the array.

VIII. Plotting of the Electric and Potential Fields in the X-Y Plane (3)

In the following plots, the electric and potential fields in the X-Y plane are depicted for periods $b = 1.1a, 1.5a, 2.5a, 4a$. The assumptions are: $\theta = \pi/4$ and $\varepsilon = 10$.







As can be seen from the plots above, the factor $S = \frac{a^2}{b^2}\frac{\pi^2}{12}$ controls all the non-locality. We observe that for small $\frac{b}{a}$, due to the inversely proportional relationship of S with $\frac{b}{a}$, we have: when $\frac{b}{a} = 1.1$, S is large, hence there is strong interaction. For sparser arrays, i.e., for $\frac{b}{a} = 1.5, 2.5$, and 4, S is small, so we revert close to the single cylinder solution.

References

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