Cylindrical Magnetostatic Cloak

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***Abstract*—This project investigates the theoretical design and physical behavior of a cylindrical magnetostatic cloak used to shield a superconducting core from a uniform transverse magnetic field. By defining the scalar magnetic potential and solving Laplace's equation across three distinct boundaries, exact analytical expressions for the magnetic field intensity are derived. A key focus is determining the optimal magnetic permeability of the outer shell required to completely eliminate external field perturbations, achieving perfect invisibility of the core. The analytical model is supported by computational MATLAB simulations that plot the vector magnetic field in the xy-plane. These visualizations confirm the mathematical findings and illustrate distinct physical phenomena, including the Meissner effect, ideal magnetic cloaking, and flux concentration, under varying material permeabilities and geometric aspect ratios. The results demonstrate that as the thickness of the shielding cloak decreases, the required optimal permeability tends toward infinity.**

***Index Terms*—Boundary Value Problem, Cylindrical Cloak, Laplace Equation, Magnetic Shielding, Magnetostatics, Meissner Effect, Permeability Optimization, Superconducting Core.**

I. Solution of the Boundary Value Problem and Determination of the Magnetic Field

Since there are no free currents, we can define the magnetic field via the scalar magnetic potential:

$$H = -\nabla V_m$$

The potential satisfies Laplace's equation:

$$\Rightarrow \nabla^2 V_m = 0$$

Due to the form of the external magnetic field applied to the cylinder: $B = B_0(\hat{r}\cos\varphi + \hat{\varphi}\sin\varphi)$, the scalar magnetic potential will have the form:

$$V_m(r,\varphi) = f(r)\cos\varphi$$

We distinguish three regions for solving the boundary value problem.

- **Region 1 - Inner cylinder** ($r < a$)

Since we are in the inner cylinder, we take into account the relative magnetic permeability μ_1. The scalar magnetic potential in the first region is given by the relation:

$$V_{m1}(r,\varphi) = -C_1 r\cos\varphi$$

- **Region 2 - Outer cylinder** ($r \in (a,b)$)

Since we are in the outer cylinder, we take into account the relative magnetic permeability μ_2. The scalar magnetic potential in the second region is given by the relation:

$$V_{m2}(r,\varphi) = -\left(C_2 r - \frac{D_2}{r}\right)\cos\varphi$$

- **Region 3 - Exterior** ($r > b$)

Since we are in the exterior region, we take into account the magnetic permeability of vacuum μ_0. In this region, both the uniform magnetic field exists, which corresponds to a potential $-H_0 r\cos\varphi$, and the perturbation due to the cylinder. Therefore, the scalar magnetic potential in the third region is given by the relation:

$$V_{m3}(r,\varphi) = -H_0 r\cos\varphi + \frac{D_3}{r}\cos\varphi$$

At the material boundaries, there is a requirement that the continuity conditions are satisfied. That is, the tangential component of H and the normal component of B must be continuous.

Therefore, at the boundary $r = a$, the following holds:

$$V_{m1}(r=a) = V_{m2}(r=a) \Rightarrow -C_1 a = -\left(C_2 a - \frac{D_2}{a}\right)$$

$$-\mu_1 \frac{\partial V_{m1}(r=a)}{\partial r} = -\mu_2 \frac{\partial V_{m2}(r=a)}{\partial r} \Rightarrow \mu_1 C_1 = \mu_2\left(C_2 + \frac{D_2}{a^2}\right)$$

and at the boundary $r = b$, the following holds:

$$V_{m2}(r=b) = V_{m3}(r=b) \Rightarrow -\left(C_2 b - \frac{D_2}{b}\right) = -H_0 b + \frac{D_3}{b}$$

$$-\mu_2 \frac{\partial V_{m2}(r=b)}{\partial r} = -\mu_0 \frac{\partial V_{m3}(r=b)}{\partial r}$$

$$\Rightarrow \mu_2\left(C_2 + \frac{D_2}{b^2}\right) = \mu_0\left(H_0 + \frac{D_3}{b^2}\right)$$

Solving the system:

$$\mu_1 \left(C_2 - \frac{D_2}{a^2} \right) = \mu_2 \left(C_2 + \frac{D_2}{a^2} \right)$$
$$(\mu_1 - \mu_2) C_2 = (\mu_1 + \mu_2) \frac{D_2}{a^2}$$
$$D_2 = a^2 \frac{(\mu_1 - \mu_2)}{(\mu_1 + \mu_2)} C_2$$

$$\mu_2 \left(C_2 + \frac{D_2}{a^2} \right) = \mu_0 \left(2H_0 - C_2 + \frac{D_2}{b^2} \right)$$
$$C_2(\mu_0 + \mu_2) + \frac{D_2}{b^2}(\mu_2 - \mu_0) = 2\mu_0 H_0$$
$$C_2(\mu_0 + \mu_2) + C_2 \frac{a^2}{b^2} \frac{(\mu_1 - \mu_2)}{(\mu_1 + \mu_2)} (\mu_2 - \mu_0) = 2\mu_0 H_0$$
$$C_2 = \frac{2\mu_0 H_0}{(\mu_0 + \mu_2) + \frac{a^2}{b^2} \frac{(\mu_1 - \mu_2)}{(\mu_1 + \mu_2)} (\mu_2 - \mu_0)}$$

We follow the same procedure to calculate D_2, D_3, and C_1, which for brevity are provided below:

$$D_2 = C_2 a^2 \frac{(\mu_1 - \mu_2)}{(\mu_1 + \mu_2)}$$
$$D_3 = b^2 (H_0 - C_2) + D_2$$
$$C_1 = C_2 \left(1 - \frac{(\mu_1 - \mu_2)}{(\mu_1 + \mu_2)} \right)$$

The magnetic field intensity vector $H = -\nabla V_m$ is given in polar coordinates by the relation:

$$H = -\frac{\partial V_m}{\partial r} \hat{r} - \frac{1}{r} \frac{\partial V_m}{\partial \varphi} \hat{\varphi}$$

Therefore, for the three regions, the magnetic field is:

- **Region 1** - $r < a$: $H_1 = C_1 (\hat{r} \cos\varphi - \hat{\varphi} \sin\varphi)$
- **Region 2** - $r \in (a, b)$: $H_2 = \left(C_2 + \frac{D_2}{r^2} \right) \cos\varphi \hat{r} - \left(C_2 - \frac{D_2}{r^2} \right) \sin\varphi \hat{\varphi}$
- **Region 3** - $r > b$: $H_3 = \left(H_0 + \frac{D_3}{r^2} \right) \cos\varphi \hat{r} - \left(H_0 - \frac{D_3}{r^2} \right) \sin\varphi \hat{\varphi}$

II. Finding and Representing the Optimal Permeability

For a superconducting core with relative magnetic permeability $\mu_1 = 0$, $B = 0$. In order not to disturb the external field, the perturbation term in Region 3 must vanish. That is, we require:

$$D_3 = 0$$

Substituting these conditions into the continuity equations, we find:

- At $r = a$, since $\mu_1 = 0$, the condition for B_r gives us:

$$0 = \mu_2 \left(C_2 + \frac{D_2}{a^2} \right) \Rightarrow D_2 = -C_2 a^2$$

- At $r = b$, since $D_3 = 0$, our equations give:

$$\text{Potential: } -\left(C_2 b - \frac{D_2}{b} \right) = -H_0 b \Rightarrow C_2 \left(b + \frac{a^2}{b} \right) = H_0 b$$
$$\text{Flux: } \mu_2 \left(C_2 + \frac{D_2}{b^2} \right) = \mu_0 H_0 \Rightarrow \mu_2 C_2 \left(1 - \frac{a^2}{b^2} \right) = \mu_0 H_0$$

Dividing the two new relations side-by-side to eliminate C_2 and H_0, we obtain the following relations:

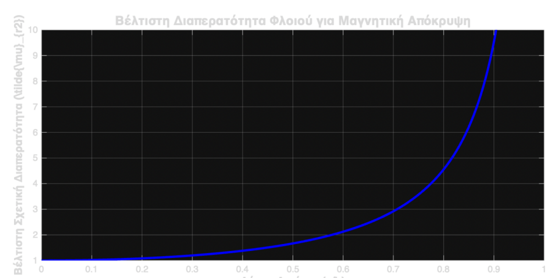
$$\frac{\mu_2 C_2 \left(1 - \frac{a^2}{b^2} \right)}{C_2 \left(b + \frac{a^2}{b} \right)} = \frac{\mu_0 H_0}{H_0 b}$$
$$\frac{\mu_2 \left(1 - \frac{a^2}{b^2} \right)}{b \left(1 + \frac{a^2}{b^2} \right)} = \frac{\mu_0}{b}$$
$$\mu_2 \frac{b^2 - a^2}{b^2 + a^2} = \mu_0$$

Thus, we can extract the formula for the optimal permeability, which is:

$$\tilde{\mu}_2 = \mu_0 \frac{b^2 + a^2}{b^2 - a^2}$$

Therefore, the material must be ferromagnetic or paramagnetic (although natural paramagnetic materials have a very weak response). For this reason, the material would ideally be an Iron Powder Composite designed to have $\tilde{\mu}_2 \approx 1.67$, since solid metals have excessively high permeability for such a "thick" outer cylinder (cloak).

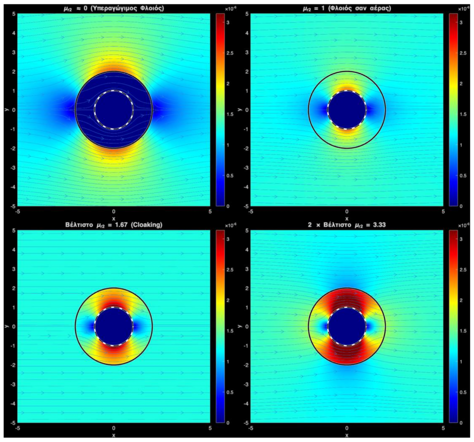
With the help of MATLAB software, the following graph was plotted, which presents the optimal permeability $\tilde{\mu}_2$ with respect to the ratio a/b.



From the graph above, we draw the following conclusions. First, the permeability $\tilde{\mu}_2$ is always greater than μ_0. Second, as the thickness of the outer cylinder decreases ($a \to b$), the required permeability tends to infinity. Finally, we understand that the role of the cloak is to "pull" the magnetic field lines into itself, guiding them around the superconductor, and to restore them to their straight path upon exiting.

III. Representation of the Vector Magnetic Field in the X-Y Plane

Below, the vector magnetic field is depicted in the xy plane, for permeability $\mu_2 = 0, 1, \tilde{\mu}_2$, and $2\tilde{\mu}_2$, under the condition $b/a = 2$.

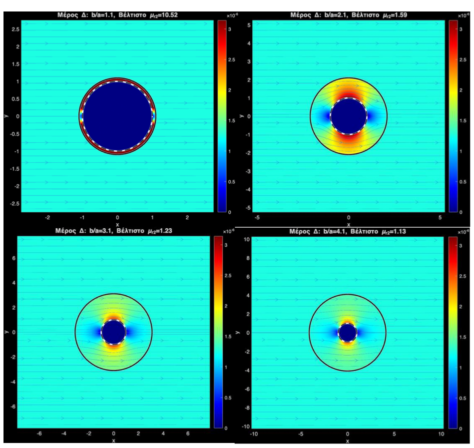


From the graphs above, we draw the following conclusions. When $\mu_2 = 0$, the outer cylinder also acts as a superconductor, completely repelling the field. Thus, the perturbation is large. When $\mu_2 = 1$, the outer cylinder acts like air. Thus, the field is perturbed solely by the inner superconductor (Meissner effect).

When $\mu_2 = \tilde{\mu}_2$, the field lines bend smoothly inside the cloak and exit parallel. Therefore, perfect cloaking is achieved. Finally, when $\mu_2 = 2\tilde{\mu}_2$, the outer cylinder "sucks in" an excessive amount of the field, creating flux concentration (focusing) instead of cloaking.

IV. Representation of the Vector Magnetic Field in the XY Plane, for the Optimal Permeability

Below, the vector magnetic field is depicted in the xy plane, for ratios $b/a = 1.1, 2.1, 3.1, 4.1$, under the condition $\mu_2 = \tilde{\mu}_2$.



From the graphs above, we draw the conclusion that the thinner the cloak is, the larger the value of magnetic permeability required to achieve cloaking.

References

[1] I. A. Roumeliotis and I. Tsalamengas, *Electromagnetic Fields, Volume II*, 2nd ed. Tziolas Publications, Thessaloniki, 2010.

[2] N. K. Uzunoglu, *Basic Principles of Electromagnetism* (in Greek), Symmetria Publications, Athens, 2004.

[3] I. A. Roumeliotis, *Electromagnetic Fields, Volume I* (in Greek), 2nd ed. Tziolas Publications, Thessaloniki, 2007.