Bistability in Electrostatic Nanopheres

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***Abstract*—This project investigates the phenomenon of bistability within electrostatic nanospheres, characterized by a nonlinear relative permittivity. By applying Gauss's law to a spherically symmetric system, we derive the equilibrium equations governing the radial electric field. Analytical conditions are established for the onset of bistability, revealing a strict dependency on the nonlinearity parameters of the material (specifically, requiring $\gamma > 8$). Furthermore, we numerically evaluate the bistability boundaries for specific material constants ($\gamma = 20$, $\kappa = 20$) and map the localized electric field values across different radial positions within the sphere's mantle. Our analysis demonstrates the emergence of characteristic hysteresis loops, driven by saddle-node bifurcations, where the internal field undergoes abrupt transitions between low- and high-intensity stable states in response to a varying external field. Finally, we explore the spatial decay of bistability outside the sphere and map the radial field distribution under various constant external supply fields, providing a comprehensive overview of the highly nonlinear electrostatic response of the system.**

***Index Terms*—Bistability, Non-Linear Electrostatics, Nanospheres, Hysteresis, Non-Linear Permittivity.**

Question A – Conditions leading to bistability

Nonlinear relative permittivity:

$$\varepsilon(E) = 1 + \frac{\gamma}{1+\kappa|E|^2} \quad \text{with } \gamma, \kappa > 0$$

From symmetry, for the radial field it holds:

$$\mathbf{E}(r) = E(r)\hat{r}$$

Thus, using Gauss's law (integral form) we obtain:

$$\oint E \cdot dS = \frac{Q}{\varepsilon_0 \varepsilon(E)} \Rightarrow 4\pi r^2 E(r) = \frac{Q}{\varepsilon_0 \varepsilon(E)} \Rightarrow$$
$$\Rightarrow E(r)\varepsilon(E) = \frac{Q}{4\pi\varepsilon_0 r^2} \Rightarrow$$
$$\Rightarrow E_0 = \frac{Q}{4\pi\varepsilon_0 a^2} \Rightarrow$$
$$\Rightarrow \frac{Q}{4\pi\varepsilon_0 r^2} = E_0\left(\frac{a}{r}\right)^2$$

Therefore, the equilibrium equation that gives $E(r)$ is the following:

$$E(r)\left(1+\frac{\gamma}{1+\kappa E(r)^2}\right) = E_0\left(\frac{a}{r}\right)^2$$

For bistability to exist, we need multiple solutions in an algebraic equation. Therefore, the function:

$$f(E) = E\left(1+\frac{\gamma}{1+\kappa E^2}\right)$$

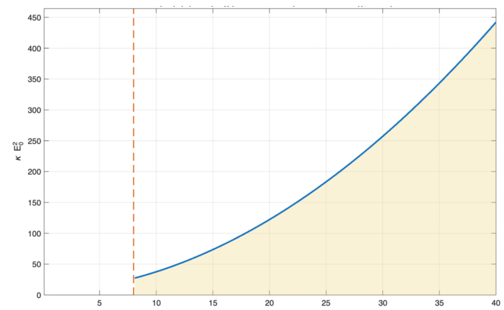


Figure 1. Bistability Area

must not be monotonic. This occurs when:

$$\frac{df}{dE} = 0 \Rightarrow$$
$$\Rightarrow \frac{df}{dE} = 1 + \frac{\gamma}{1+\kappa E^2} - \frac{2\gamma\kappa E^2}{(1+\kappa E^2)^2}$$

We set:

$$\frac{df}{dE} = 0 \Rightarrow 1 + \frac{\gamma(1-\kappa E^2)}{(1+\kappa E^2)^2} = 0$$

In order for the equation:

$$\frac{df}{dE} = 0 \Rightarrow 1 + \frac{\gamma(1-\kappa E^2)}{(1+\kappa E^2)^2} = 0$$

to have solutions, $f(E)$ must acquire inflection points.

$$\Rightarrow \gamma > 8$$

Therefore, in the two-dimensional plane $(\gamma, \kappa E_0^2)$, the following hold:

- For $\gamma \leq 8 \Rightarrow$ Monotonicity $\Rightarrow$ Bistability does not exist.
- For $\gamma > 8 \Rightarrow$ Lack of monotonicity $\Rightarrow$ Bistability exists.

Below, the bistability region is shown in the $\gamma, \kappa E_0^2$ plane, in a graph generated using MATLAB.

Question B – Value ranges for bistability

The function $f(E)$ with $\gamma = 20$, $\kappa = 20$ presents two critical values E_1 and E_2 where $df/dE = 0$. Calculating them, we obtain:

$$E_1 \approx 0.287 \text{ V/m and } E_2 \approx 0.604 \text{ V/m}$$

They correspond to values $f(E_1), f(E_2)$:

$$f(E_1) \approx 0.935, f(E_2) \approx 0.762$$

(units V/m, with normalization $E_0 = 1$).

Therefore, for constant r there are three solutions when

$$f_{\min} < S < f_{\max},$$

$$\Rightarrow f_{\min} = 1.9463055 < S < f_{\max} = 2.4722247.$$

$S = E_0(a/r)^2 \Rightarrow$ Therefore the interval of values of E_0 for which bistability exists, for a given r, is

$$E_{0,\min}(r) = f_{\min}\left(\frac{r}{a}\right)^2, E_{0,\max}(r) = f_{\max}\left(\frac{r}{a}\right)^2.$$

Consequently, the interval for E_0^2 is simply the squares of these limits.

In the graph below, the parametric region that leads to bistability is depicted.

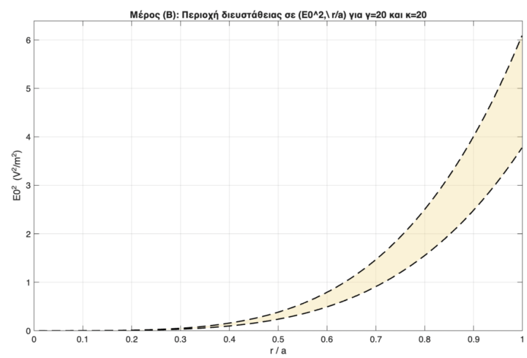


Figure 2. Bistability Area

The field outside the sphere, i.e., for some point where $r > a$ holds, exhibits bistability only when:

$$E_0\left(\frac{r}{a}\right)^2 \in [f_{\min}, f_{\max}]$$

where $f_{\min}$ and $f_{\max}$ are the values of $f(E)$ at the local extrema (where $f'(E) = 0$).

Consequently, for $r > a$, bistability occurs only if $E_0 < \frac{f_{\max}}{\left(\frac{a}{r}\right)^2}$ and $E_0 > \frac{f_{\min}}{\left(\frac{a}{r}\right)^2}$. Therefore, the further away from the sphere one moves, the more difficult it is to maintain bistability, as the field tends to become monostable because $E_0(a/r)^2$ decreases.

Question C – Possible Electric Field Values for 4 Specific Mantle Positions

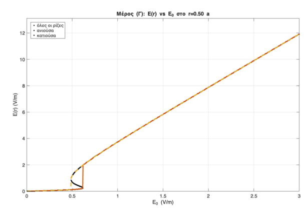
For constant positions inside the mantle ($r = 0.50a, 0.60a, 0.75a, 0.95a$), we solve the equation:

$$f(E) = E_0\left(\frac{a}{r}\right)^2$$

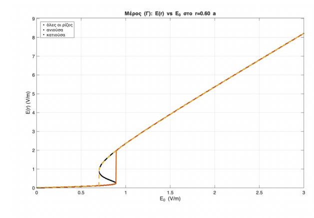
from which we obtain the following values:

- $r = 0.50a$
  - $E_0 \in [0.486576, 0.618056]$
  - $E_0^2 \in [0.236757, 0.381993]$
- $r = 0.60a$
  - $E_0 \in [0.700670, 0.890001]$
  - $E_0^2 \in [0.490938, 0.792102]$
- $r = 0.75a$
  - $E_0 \in [1.094797, 1.390626]$
  - $E_0^2 \in [1.198580, 1.933842]$
- $r = 0.95a$
  - $E_0 \in [1.756541, 2.231183]$
  - $E_0^2 \in [3.085435, 4.978177]$

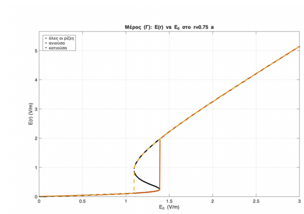
Below, the graphs for the four points of the spherical mantle are presented.



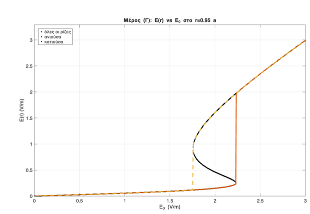
(a) $r = 0.50a$



(b) $r = 0.60a$



(c) $r = 0.75a$



(d) $r = 0.95a$

Figure 3. Graphs depicting the electric field values for the four specific positions of the spherical mantle.

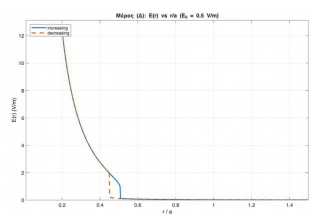
From the above graphs, we conclude that for small values of the external field E_0, i.e., outside the bistability intervals, there is a single, unique solution and the internal field increases monotonically with E_0. When E_0 enters the interval where three solutions exist, three possible regimes appear: a low-intensity stable solution, an intermediate solution (usually unstable), and a high-intensity stable solution.

During the upward path (solid line), the system follows the low solution until the tipping point (saddle-node), where it suddenly "jumps" to the high solution. This manifests as an abrupt jump in the graph. Conversely, during the downward path (dashed line), the system remains in the high solution until it reaches the second critical point, where it suddenly drops to the low one. The result is a characteristic hysteresis loop.

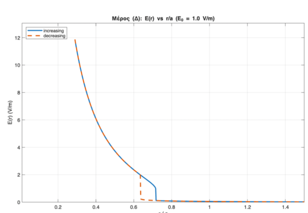
In general, the closer to the surface of the sphere, the more the bistability interval shifts towards larger values of E_0 and widens.

Question D – Electric field along the radius of the sphere, with a constant supply field

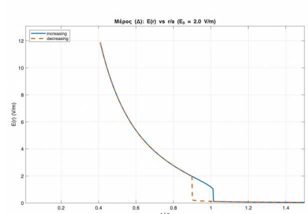
For $\gamma = 20$ and $\kappa = 20$ (m/V)^2, the electric field was plotted along the radius of the sphere (and slightly outside it: $0 < r/a < 1.5$), keeping the supply field E_0 constant. Four constant excitation values were examined: $E_0 = 0.5$ V/m, $E_0 = 1$ V/m, $E_0 = 2$ V/m and $E_0 = 3$ V/m in separate diagrams, which are shown below.



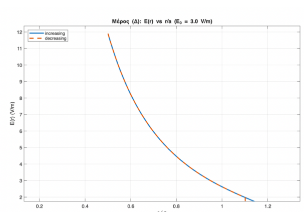
(a) $E_0 = 0.5$ V/m



(b) $E_0 = 1$ V/m



(c) $E_0 = 2$ V/m



(d) $E_0 = 3$ V/m

Figure 4. Graphs depicting the electric field along the radius for constant supply field values.

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